WPI Mathematical Sciences Department
Ph.D. General Comprehensive Exam, MA 541
January, 2013

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1. Suppose $X_1, \dots, X_n$ is a random sample from a $U(\theta - 1/2, \theta + 1/2)$ distribution. Characterize the set of all maximum likelihood estimators of θ.

2. Suppose k independent samples are taken and for each sample a test of $H_0 : \theta = \theta_0$ is performed. Let T_i denote the test statistic and $\alpha(T_i)$ the p-value of the test from sample i. The statistic $V = -2\sum_{i=1}^{k} \ln[\alpha(T_i)]$ was recommended by Fisher as a way to combine all these tests into a single test. What is the distribution of V if H_0 is true?

3. A single observation X is taken from a distribution with density f. It is desired to test the hypotheses $H_0 : f = f_0$ versus $H_a : f = f_a$, where f_0 is a $U(0, 1)$ pdf and f_a is the triangular pdf on [0,1]:

$$\begin{aligned} f_a(x) &= 4x,\ 0 \le x < 1/2, \\ &= 4 - 4x,\ 1/2 \le x \le 1. \end{aligned}$$

   Construct a most powerful level α test of H_0 versus H_a.

4. Suppose that W is the UMVUE of an unknown parameter θ, and that it has moments of all orders. Show that W^k is a UMVUE of $E(W^k)$, where k is any positive integer.

5. Let $\boldsymbol{X} = (X_1, \dots, X_n)$ be a random sample from a population with mean μ and variance σ^2, both of which are unknown. Let $T(\boldsymbol{X}) = \sum_{i=1}^{n} c_i X_i$ be an estimator of μ. Which values of the c_i give an unbiased estimator with minimum variance among all unbiased estimators of this form?

6. Consider a random sample from a bivariate normal distribution,

$$\boldsymbol{X}_1, \dots, \boldsymbol{X}_n \sim N(\boldsymbol{\mu}, \Sigma),$$

   where

$$\boldsymbol{X}_i' = [X_{1i}, X_{2i}],\ i = 1 \dots, n,\ \boldsymbol{\mu}' = [\mu_1, \mu_2],\ \text{and } \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$$

   (a) Suppose $\mu_2 \neq 0$ and define $r = \mu_1/\mu_2$. Find the distribution of $\overline{X}_1 - r\overline{X}_2$, where $\overline{X}_1$ and $\overline{X}_2$ are the sample means of the first and second variates, respectively.

   (b) Use the result in (a) to construct a level $1 - \alpha$ confidence interval for r.

7. Suppose for each $n = 1, 2, ..., X_1, \dots, X_n$ is a random sample from a distribution $f(x|\theta),\ \theta \in \Theta \subset \mathbb{R}$. Let $l_n(\theta)$ denote the log likelihood. Assume the standard set of regularity conditions so that under $H_0 : \theta = \theta_0$, the quantities below converge in the standard way.

   (a) $n^{-1} l_n''(\theta_0)$ converges in probability. To what does it converge? Be as specific as you can.

   (b) $\sqrt{n}(\hat{\theta}_n - \theta_0)$ converges in distribution, where $\hat{\theta}_n$ is the MLE. To what distribution does it converge? Use the result from (a) and be as specific as you can.

   (c) $n^{-1/2} l_n'(\theta_0)$ converges in distribution. To what distribution does it converge? Use the result from (a) and be as specific as you can.